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A Topological Category Suited for Approximation Theory?

R. LOWEN

Dienst Wiskundige Analyse, Groenenborgerlaan 171, 2020 Antwerpen, Belgium

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This paper presents a framework in which basic concepts of approximation theory arise as canonically as convergence in topological spaces. © 1989 Academic Press, Inc.

1. INTRODUCTION

Approximation theory widely uses concepts and techniques from at least two canonical mathematical structures: the theory of topological spaces and the isometric theory of metric and normed spaces. Given a metric space (X, d), to say $(x_n)_n$ converges to x is a topological concept, to approximate x by x_n depending on "how small" $d(x, x_n)$ is, is a metric concept. To say x is in the closure of A is a topological concept, to approximate x with a "best possible" element in A based on $d(x, A) := \inf\{d(x, a) \mid a \in A\}$ is a metric concept.

Moreover, in both cases the "topological situation" is in a sense the "best metric situation."

Neither canonical topological nor metric concepts exist to deal with the difference between the non-convergent sequences $(\varepsilon(-1)^n)_n$ and $(n(-1)^n)_n$ in $\mathbb{R}$ whereas, depending on the actual application, the former might be considered "approximately constant" and thus "approximately convergent."

These observations, however simple, lead us to wonder whether there exists a framework in which "approximation–convergence" could be developed in the same way as, e.g., convergence is developed in TOP (the category of topological spaces and continuous maps).

It is the purpose of this paper to question whether AP (the category of approach spaces and contractions) [6] might not be a nice and relatively easy framework suited to this end.

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2. Preliminaries

We recall those concepts of [6] which are required in the sequel. Let X be an arbitrary set, $\overline{\mathbb{R}}_+ := [0, \infty]$, and $\mathbb{R}_+^* :=]0, \infty[$. A map

$$\delta: X \times 2^X \to \overline{\mathbb{R}}_+$$

is called a *distance* if it fulfils

(D1) $\forall A \in 2^X, \forall x \in X: x \in A \Rightarrow \delta(x, A) = 0$,

(D2) $\forall x \in X: \delta(x, \phi) = \infty$,

(D3) $\forall A, B \in 2^X, \forall x \in X: \delta(x, A \cup B) = \delta(x, A) \wedge \delta(x, B)$,

(D4) $\forall A \in 2^X, \forall x \in X, \forall \varepsilon \in \overline{\mathbb{R}}_+: \delta(x, A) \leqslant \delta(x, A^{(\varepsilon)}) + \varepsilon$ where $A^{(\varepsilon)} := \{x \mid \delta(x, A) \leqslant \varepsilon\}$.

A collection $(\Phi(x))_{x \in X}$ of ideals in $\overline{\mathbb{R}}_+^X$ is called an *approach system* if it fulfils

(A1) $\forall x \in X, \forall v \in \Phi(x): v(x) = 0$,

(A2) $\forall x \in X, \forall v \in \overline{\mathbb{R}}_+^X: \forall \varepsilon, N \in \mathbb{R}_+^*, \exists v_\varepsilon^N \in \Phi(x)$ s.t. $v_\varepsilon^N + \varepsilon \geqslant v \wedge N \Rightarrow v \in \Phi(x)$,

(A3) $\forall x \in X, \forall v \in \Phi(x), \forall N \in \mathbb{R}_+^*, \exists (v_z)_{z \in X} \in \prod_{x \in X} \Phi(x), \forall z, y \in X: v_x(z) + v_z(y) \geqslant v(y) \wedge N$.

Each ideal $\Phi(x)$ is called an *approach ideal* (at x) and the functions in $\Phi(x)$ are called *neighborhood maps* (at x). For ease in notation we shall, whenever convenient denote an approach system $(\Phi(x))_{x \in X}$ also simply Φ.

In [6] we showed that distances and approach systems are equivalent to each other, and we gave the following formulas expressing one concept in terms of the other. Given a distance δ, the associated approach system Φ_δ is given by

$$\Phi_\delta(x) = \{v \in \overline{\mathbb{R}}_+^X \mid \forall A \subset X: \inf_{a \in A} v(a) \leqslant \delta(x, A)\}, \qquad x \in X,$$

and given an approach system Φ, the associated distance δ_Φ is given by

$$\delta_\Phi(x, A) = \sup_{v \in \Phi(x)} \inf_{a \in A} v(a), \qquad x \in X, A \subset X.$$

Moreover it was shown that $\Phi_{\delta_\Phi} = \Phi$ and that $\delta_{\Phi_\delta} = \delta$. If no confusion can occur we write Φ (resp. δ) instead of Φ_δ (resp. δ_Φ).

A set X equipped with an approach system (or equivalently a distance) is called an *approach space* and is usually denoted (X, Φ) or simply X if no confusion can occur.

If X and X' are approach spaces then a map $f: X \to X'$ is called a *contraction* if the following equivalent conditions are fulfilled:

(C1) $\forall x \in X, \forall v' \in \Phi'(f(x)): v' \circ f \in \Phi(x)$,

(C2) $\forall x \in X, \forall A \subset X: \delta'(f(x), f(A)) \leqslant \delta(x, A)$.

Approach spaces and contractions form a topological category [2] which we denote AP. TOP is embedded bireflectively and bicoreflectively by

$$\begin{array}{ccc} \mathrm{TOP} & \xrightarrow{A_t} & \mathrm{AP} \\ (X, \mathbf{T}) & \longrightarrow & (X, A_t(\mathbf{T})), \end{array}$$

where $A_t(\mathbf{T})(x) := \{v \mid v(x) = 0,\ v \text{ u.s.c. at } x\}$ for all $x \in X$. The associated distance is given by $\delta(x, A) = 0$ iff $x \in \bar{A}$ and $\delta(x, A) = \infty$ iff $x \notin \bar{A}$ for all $x \in X$, $A \subset X$.

Given $(X, \Phi) \in |\mathrm{AP}|$ its TOP-coreflection is given by

$$(X, A_t(T^*(\Phi))) \xrightarrow{\mathrm{id}_X} (X, \Phi),$$

where $T^*(\Phi)$ is the topology determined by the neighborhood system

$$N^*(\Phi)(x) := \{\{v < \varepsilon\} \mid v \in \Phi(x), \varepsilon \in \mathbb{R}_+^*\}, \qquad x \in X.$$

T^* is of course left inverse, right adjoint to A_t. Analogously $p-q$-MET$^\infty$ ($\infty-p-q$-metric spaces, non-expansive maps) is embedded bicoreflectively by

$$\begin{array}{ccc} p-q\text{-MET}^\infty & \xrightarrow{A_m} & \mathrm{AP} \\ (X, d) & \longrightarrow & (X, A_m(d)), \end{array}$$

where $A_m(d)(x) := \{v \mid v \leqslant d(x, \cdot)\}$ for all $x \in X$. The associated distance in this case is given by $\delta(x, A) = \inf_{a \in A} d(x, a)$ for all $x \in X$, $A \subset X$.

Given $(X, \Phi) \in |\mathrm{AP}|$ its $p-q$-MET$^\infty$-coreflection is given by

$$(X, A_m \circ M(\Phi)) \xrightarrow{\mathrm{id}_X} (X, \Phi),$$

where $M(\Phi)$ is the $\infty-p-q$-metric defined by $M(\Phi)(x, y) := \delta_\Phi(x, \{y\})$. M is of course left inverse, right adjoint to A_m.

3. Convergence in AP

For a more detailed study we refer the reader to [5, 7]. When dealing with metric spaces we shall sometimes "translate" the general results for filters to sequences. For the sequel of this section, unless otherwise specified let $(X, \Phi) \in |\mathrm{AP}|$. For a filter $\mathbf{F}$ on X we define the maps

$$\alpha\mathbf{F}(x) := \sup_{\nu \in \Phi(x)} \sup_{F \in \mathbf{F}} \inf_{y \in F} \nu(y), \qquad x \in X$$

$$= \sup_{F \in \mathbf{F}} \delta(x, F)$$

$$\lambda\mathbf{F}(x) := \sup_{\nu \in \Phi(x)} \inf_{F \in \mathbf{F}} \sup_{y \in F} \nu(y), \qquad x \in X.$$

For each filter $\mathbf{F}$ on X, $\alpha\mathbf{F}$ (resp. $\lambda\mathbf{F}$) "measures" the adherence (resp. convergence) of $\mathbf{F}$. This shall become clear in the sequel. A first indication is given by the following result, the easy verification of which is left to the reader. Lim $\mathbf{F}$ and adh $\mathbf{F}$ stand for the set of limit (resp. adherence) points of $\mathbf{F}$.

PROPOSITION 3.1. *If $(X, \mathbf{T})$ is a topological space and $\mathbf{F}$ a filter on X, then in $(X, A_t(\mathbf{T}))$ we have*

$$\alpha\mathbf{F}(x) = \begin{cases} 0, & x \in \operatorname{adh} \mathbf{F} \text{ in } (X, \mathbf{T}) \\ \infty, & \text{otherwise} \end{cases}$$

and

$$\lambda\mathbf{F}(x) = \begin{cases} 0, & x \in \lim \mathbf{F} \text{ in } (X, \mathbf{T}) \\ \infty, & \text{otherwise.} \end{cases}$$

A few elementary facts hold for topological convergence. For example, a limit point of a filter is an adherence point, if $\mathbf{F} \subset \mathbf{G}$ are filters then adherence points of $\mathbf{G}$ are adherence points of $\mathbf{F}$ a.s.o.

The following results are the analogons in AP. Verifications are left to the reader.

PROPOSITION 3.2. *For filters $\mathbf{F} \subset \mathbf{G}$ we have*

$$\alpha\mathbf{F} \leqslant \alpha\mathbf{G} \leqslant \lambda\mathbf{G} \leqslant \lambda\mathbf{F}.$$

PROPOSITION 3.3. *If $\mathbf{U}$ is an ultrafilter then $\alpha\mathbf{U} = \lambda\mathbf{U}$.*

If $\mathbf{F}$ is a filter then we shall denote $U(\mathbf{F})$ the set of all ultrafilters finer than $\mathbf{F}$. In a topological space a filter converges to a point (resp. adheres to a point) if and only if all ultrafilters finer adhere to that point (resp. some ultrafilter finer adheres to that point). The following result is the analogon in AP.

PROPOSITION 3.4. *For a filter* **F** *the following hold*:

(1) $\lambda \mathbf{F} = \sup_{\mathbf{U} \in U(\mathbf{F})} \alpha \mathbf{U}$,

(2) $\alpha \mathbf{F} = \inf_{\mathbf{U} \in U(\mathbf{F})} \alpha \mathbf{U}$.

Proof. (1) By Proposition 3.3 and applying complete distributivity we obtain

$$\begin{aligned}\sup_{\mathbf{U} \in U(\mathbf{F})} \alpha \mathbf{U}(x) = \sup_{\mathbf{U} \in U(\mathbf{F})} \lambda \mathbf{U}(x) & \\ &= \sup_{v \in \Phi(x)} \inf_{\phi \in \prod_{\mathbf{U} \in U(\mathbf{F})} \mathbf{U}} \sup_{\mathbf{U} \in U(\mathbf{F})} \sup_{y \in \phi(\mathbf{U})} v(y) \\ &= \sup_{v \in \Phi(x)} \inf_{\phi \in \prod_{\mathbf{U} \in U(\mathbf{F})} \mathbf{U}} \sup_{y \in \bigcup_{\mathbf{U} \in U(\mathbf{F})} \phi(\mathbf{U})} v(y) \\ &= \sup_{v \in \Phi(x)} \inf_{F \in \mathbf{F}} \sup_{y \in F} v(y) \\ &= \lambda \mathbf{F}(x).\end{aligned}$$

(2) Applying complete distributivity we obtain

$$\begin{aligned}\inf_{\mathbf{U} \in U(\mathbf{F})} \alpha \mathbf{U}(x) = \inf_{\mathbf{U} \in U(\mathbf{F})} \sup_{U \in \mathbf{U}} \delta(x, U) & \\ &= \sup_{\phi \in \prod_{\mathbf{U} \in U(\mathbf{F})} \mathbf{U}} \inf_{\mathbf{U} \in U(\mathbf{F})} \delta(x, \phi(\mathbf{U})). \qquad (*)\end{aligned}$$

Claim. $\forall \phi \in \prod_{\mathbf{U} \in U(\mathbf{F})} \mathbf{U}\ \exists U_\phi \subset U(\mathbf{F})$ finite: $\bigcup_{\mathbf{U} \in U_\phi} \phi(\mathbf{U}) \in \mathbf{F}$. Indeed, if not, then for each finite $U_0 \subset U(\mathbf{F})$ we have

$$\bigcup_{\mathbf{U} \in U_0} \phi(\mathbf{U}) \notin \mathbf{F},$$

and then the family

$$\mathbf{F} \cup \{X \backslash \phi(\mathbf{U}) \mid \mathbf{U} \in U(\mathbf{F})\}$$

has the finite intersection property and is contained in some ultrafilter $\mathbf{U}_0 \in U(\mathbf{F})$. This, however, is impossible, since then $X \backslash \phi(\mathbf{U}_0) \in \mathbf{U}_0$. Thus from (*), our claim, and (D3) we obtain

$$\begin{aligned}\inf_{\mathbf{U} \in U(\mathbf{F})} \alpha \mathbf{U}(x) \leqslant \sup_{\phi \in \prod_{\mathbf{U} \in U(\mathbf{F})} \mathbf{U}} \inf_{\mathbf{U} \in U_\phi} \delta(x, \phi(\mathbf{U})) & \\ &= \sup_{\phi \in \prod_{\mathbf{U} \in U(\mathbf{F})} \mathbf{U}} \delta\left(x, \bigcup_{\mathbf{U} \in U_\phi} \phi(\mathbf{U})\right) \\ &\leqslant \sup_{F \in \mathbf{F}} \delta(x, F) = \alpha \mathbf{F}(x).\end{aligned}$$

The other inequality follows at once from Proposition 3.2. ∎

4. AP-Convergence in Metric Spaces

If $(X, d) \in |\mathrm{MET}|$, $x \in X$, and $A \subset X$ then we denote $d(x, A)$ the usual distance from x to A and $d(A)$ the diameter of A.

If $\mathbf{F}$ is a filter on X, we define its *width* as

$$w(\mathbf{F}) := \inf_{F \in \mathbf{F}} d(F).$$

If $(x_n)_n$ is a sequence in X we denote $\langle x_n \rangle$ the Fréchet filter generated by it. Clearly $w(\langle x_n \rangle) = \inf_{n \in \mathbb{N}} \sup_{k,l \geqslant n} d(x_k, x_l)$. The next result follows at once from the definitions and the nature of the embedding A_m.

Proposition 4.1. *If (X, d) is a metric space, $x \in X$, and $\mathbf{F}$ is a filter on X then in $(X, A_m(d))$ we have*:

(1) $\alpha\mathbf{F}(x) = \sup_{F \in \mathbf{F}} d(x, F)$,

(2) $\lambda\mathbf{F}(x) = \inf_{F \in \mathbf{F}} \sup_{y \in F} d(x, y)$.

If $(x_n)_n$ is a sequence one easily verifies that Proposition 4.1 implies $\alpha(\langle x_n \rangle)(x) = \underline{\lim}\, d(x, x_n)$ and $\lambda(\langle x_n \rangle)(x) = \overline{\lim}\, d(x, x_n)$.

Proposition 4.2. *If (X, d) is a metric space, $x \in X$, and $\mathbf{F}$ is a filter on X then the following are equivalent*:

(1) $\mathbf{F} \to x$ *in* (X, d),

(2) $\lambda\mathbf{F} = d(\cdot, x) = d(\cdot, \lim \mathbf{F})$ *in* $(X, A_m(d))$.

Proof. (1) $\Rightarrow$ (2). From Proposition 4.1 for any $x, y \in X$ we have

$$\begin{aligned} \lambda\mathbf{F}(y) &\leqslant \inf_{F \in \mathbf{F}} \sup_{z \in F} (d(y, x) + d(x, z)) \\ &= d(y, x) + \lambda\mathbf{F}(x) = d(y, x) \end{aligned}$$

and conversely for any $\varepsilon > 0$ and $F \in \mathbf{F}$ taking $z \in B(x, \varepsilon) \cap F$ we obtain

$$\lambda\mathbf{F}(y) + \varepsilon \geqslant \inf_{F \in \mathbf{F}} d(y, z) + d(z, x) \geqslant d(y, x).$$

(2) $\Rightarrow$ (1). From the description of $T^*(\Phi)$ with $\Phi = A_m(d)$ it follows that $\lambda\mathbf{F}(x) = 0$ implies $\mathbf{F} \to x$. ∎

The smaller the value of $\lambda\mathbf{F}(x)$ (resp. $\alpha\mathbf{F}(x)$) the better x approximates the concept of being a limit (resp. adherence) point of $\mathbf{F}$. Consequently it is interesting to give some universal bounds on these values.

THEOREM 4.3. *If* (X, d) *is a metric space and* $\mathbf{F}$ *is a filter on* X *then in* $(X, A_m(d))$ *we have*

$$\tfrac{1}{2}w(\mathbf{F}) \leqslant \lambda\mathbf{F} \leqslant \alpha\mathbf{F} + w(\mathbf{F}).$$

Proof. To prove the first inequality let $x \in X$, $F \in \mathbf{F}$, and $\varepsilon \in \mathbb{R}_+^*$ and let $z, y \in F$ be such that $d(z, y) + 2\varepsilon \geqslant d(F)$. Then since $d(x, y) \vee d(x, z) \geqslant \frac{1}{2}d(z, y)$ we obtain

$$\tfrac{1}{2}d(F) \leqslant \sup_{t \in F} d(x, t) + \varepsilon$$

and the result follows from Proposition 4.1(2).

In order to prove the second inequality let $x \in X$, $\varepsilon \in \mathbb{R}_+^*$, and let $F_\varepsilon \in \mathbf{F}$ be such that $d(F_\varepsilon) \leqslant w(\mathbf{F}) + \varepsilon$. For all $y, z \in F_\varepsilon$ we then have

$$d(x, y) \leqslant d(x, z) + w(\mathbf{F}) + \varepsilon$$

and thus also

$$\sup_{y \in F_\varepsilon} d(x, y) \leqslant d(x, F_\varepsilon) + w(\mathbf{F}) + \varepsilon$$

and the result follows from Proposition 4.1(1) and (2). ∎

COROLLARY 4.4. *If* (X, d) *is a metric space and* $\mathbf{F}$ *is a Cauchy filter on* X *then in* $(X, A_m(d))$ *we have* $\lambda\mathbf{F} = \alpha\mathbf{F}$.

In case $\mathbf{F}$ is a total filter the foregoing results can be improved upon. Recall that $\mathbf{F}$ is called total [8, 9] if all $\mathbf{U} \in U(\mathbf{F})$ are convergent.

THEOREM 4.5. *If* (X, d) *is a metric space and* $\mathbf{F}$ *is a total filter on* X *then in* $(X, A_m(d))$ *we have*

(1) $\frac{1}{2}d(\text{adh}\,\mathbf{F}) \leqslant \lambda\mathbf{F} \leqslant \alpha\mathbf{F} + d(\text{adh}\,\mathbf{F})$,

(2) $\alpha\mathbf{F} = d(\cdot, \text{adh}\,\mathbf{F})$,

(3) $\lambda\mathbf{F} = \sup_{z \in \text{adh}\,\mathbf{F}} d(\cdot, z)$.

Proof. It follows from [9] that for any $\varepsilon \in \mathbb{R}_+^*$ we have $(\text{adh}\,\mathbf{F})^{(\varepsilon)} \in \mathbf{F}$. Since also $d((\text{adh}\,\mathbf{F})^{(\varepsilon)}) \leqslant d(\text{adh}\,\mathbf{F}) + 2\varepsilon$ it follows that for a total filter $\mathbf{F}$: $w(\mathbf{F}) = d(\text{adh}\,\mathbf{F})$. Thus (1) follows at once from Theorem 4.3.

For (2) notice that if $d(x, \text{adh}\,\mathbf{F}) < \delta$, i.e., there exists $y \in \bigcap_{F \in \mathbf{F}} \bar{F}$ such that $d(x, y) < \delta$, then $\alpha\mathbf{F}(x) \leqslant \sup_{F \in \mathbf{F}} d(x, F) \leqslant \delta$ which proves the inequality $\leqslant$. The converse inequality again follows from the totality of $\mathbf{F}$. Indeed, given $\varepsilon \in \mathbb{R}_+^*$ we have

$$\begin{aligned} d(x, \text{adh}\,\mathbf{F}) &\leqslant d(x, (\text{adh}\,\mathbf{F})^{(\varepsilon)}) + \varepsilon \\ &\leqslant \alpha\mathbf{F}(x) + \varepsilon. \end{aligned}$$

Finally (3) follows from Propositions 3.3, 3.4(1) and 4.2, and the fact that each $\mathbf{U} \in U(\mathbf{F})$ converges to some point of adh $\mathbf{F}$. ∎

Unlike Proposition 4.2(2), Proposition 4.5(2) depends on the totality of $\mathbf{F}$. If $X := \mathbb{R} \setminus \{1\}$ and $\mathbf{F}$ is the Fréchet filter generated by the sequence

$$x_n := \begin{cases} 1 - 1/n, & n \text{ odd} \\ 2, & n \text{ even,} \end{cases}$$

then adh $\mathbf{F} = \{2\}$, $\alpha\mathbf{F}(0) = 1$ but $d(0, \{2\}) = 2$.

Finally, we would like to illustrate how the bounds on $\lambda\mathbf{F}$ can still be improved upon and how the "better convergence points" can be found in case X is a Hilbert space.

If X is a normed space we denote co(A) the convex hull of a subset $A \subset X$. Then from the fact that for any $A \subset X$: $d(\text{co}(A)) = d(A)$ and Theorem 4.5 we immediately obtain that if $\mathbf{F}$ is total and $x \in \text{co}(\text{adh}\,\mathbf{F})$ then $\lambda\mathbf{F}(x) \leqslant d(\text{adh}\,\mathbf{F})$. In a Hilbert space any point x can be improved upon by a "better convergence point" in co(adh $\mathbf{F}$).

THEOREM 4.6. *If X is a Hilbert space and $\mathbf{F}$ is total then for each $x \in X$ there exists $x^* \in \text{co}(\text{adh}\,\mathbf{F})$ such that $\lambda\mathbf{F}(x^*) \leqslant \lambda\mathbf{F}(x)$.*

Proof. Since $\mathbf{F}$ is total, it follows from [9] that adh $\mathbf{F}$, and thus co(adh $\mathbf{F}$), is compact. For any $x \in X$ now let x^* be its projection on co(adh $\mathbf{F}$) and apply Theorem 4.5(3). ∎

The previous two results already show that "best convergence" is achieved on co(adh $\mathbf{F}$) and that on this set an upper bound is given by $d(\text{adh}\,\mathbf{F})$. However an old theorem of Jung [3] still improves this result.

THEOREM 4.7. *If X is a Hilbert space of dimension n (i.e., essentially $\mathbb{R}^n$) and $\mathbf{F}$ is total then there exists $x \in \text{co}(\text{adh}\,\mathbf{F})$ such that*

$$\lambda\mathbf{F}(x) \leqslant \left(\frac{n}{2(n+1)}\right)^{1/2} d(\text{adh}\,\mathbf{F})$$

and if X is infinite dimensional there exists $x \in \text{co}(\text{adh}\,\mathbf{F})$ such that

$$\lambda\mathbf{F}(x) \leqslant \frac{1}{\sqrt{2}}\, d(\text{adh}\,\mathbf{F}).$$

Proof. If X has dimension n, Jung's theorem [3] says we can find a ball B with radius less than $(n/2(n+1))^{1/2}\, d(\text{adh}\,\mathbf{F})$ such that adh $\mathbf{F} \subset B$. Since we can take the centre x of this ball in co(adh $\mathbf{F}$) the result follows again upon applying Theorem 4.5(3).

If X is infinite dimensional we reason as follows. Since adh $\mathbf{F}$ is compact, for each $n \in \mathbb{N}_0$, we can find $A_n \subset \text{adh } \mathbf{F}$ finite such that

$$\text{adh } \mathbf{F} \subset \bigcup_{a \in A_n} B\left(a, \frac{1}{n}\right). \tag{$*$}$$

Let $X_n \subset X$ be a finite-dimensional subspace of X containing A_n with dimension $m(n)$. Then again by Jung's theorem we can find $x_n \in \text{co}(A_n) \subset X_n$ such that

$$\begin{aligned} A_n \subset B\left(x_n, \left(\frac{m(n)}{2(m(n)+1)}\right)^{1/2} d(A_n)\right) \\ \subset B\left(x_n, \frac{1}{\sqrt{2}} d(A_n)\right). \end{aligned}$$

Then it follows from $(*)$ that

$$\text{adh } \mathbf{F} \subset B\left(x_n, \frac{1}{n} + \frac{1}{\sqrt{2}} d(\text{adh } \mathbf{F})\right).$$

Since $\text{co}(A_n) \subset \text{co}(\text{adh } \mathbf{F})$ and the latter set is compact we can find a subsequence $(x_{k_n})_n$ which converges to some $x \in \text{co}(\text{adh } \mathbf{F})$. Then for any $y \in \text{adh } \mathbf{F}$ and $n \in \mathbb{N}_0$ we have

$$\|x - y\| \leqslant \|x - x_{k_n}\| + \frac{1}{k_n} + \frac{1}{\sqrt{2}} d(\text{adh } \mathbf{F})$$

and the result follows letting $n \to \infty$ and once more applying Theorem 4.5(3). ∎

All foregoing results are in general best possible. For Theorems 4.3 and 4.5 this is easily verified considering $X := \mathbb{R}$ and the filter generated by the sequence $(x_n)_n$ where $x_n := a$ if n is odd, $x_n := b$ if n is even, and $a \neq b$. That the result of Theorem 4.7 is best possible follows from the fact that Jung's theorem is best possible.

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